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FINITE REPRESENTATION OF THE SOLAR GRAVITATIONAL FIELD IN FLAT SPACE OF SIX DIMENSIONS.

BY EDWARD KASNER.

In the preceding paper it was shown that no solution of the Einstein equations $G_{ik} = 0$ can represent a four-spread imbedded in a five-flat (except in the trivial case where there is no permanent gravitation, so that the four-spread is itself flat). It follows that the solar field can only be imbedded in a flat space of more than five dimensions. By the general theory of quadratic forms in four variables, the maximum number of dimensions required is ten. We shall now show that *the requisite dimensionality for the solar field is actually six.** The finite equations, in six cartesian coordinates, of the curved four-spread representing the solar field are given. As seen in formulas (7) below, a hyperelliptic integral is involved. This spread may be described as a *geometric model of the exact field* in which we are living.

The field may be taken in the usual Schwarzschild form

$$ds^2 = \gamma dt^2 - \gamma^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2, \tag{1}$$

where

$$\gamma = 1 - \frac{2m}{r};$$

here m is the mass of the sun, r, θ, φ are polar coordinates, and t is the time.

Introducing cartesian coordinates, x, y, z, we have the equivalent form

$$ds^2 = - dx^2 - dy^2 - dz^2 - \frac{2m}{r - 2m} dr^2 + \frac{r - 2m}{r} dt^2, \tag{2}$$

where

$$r = \sqrt{x^2 + y^2 + z^2}. \tag{2'}$$

In (1) we have four square terms all with variable coefficients. In (2) we have three squares with unit coefficients, and two squares with variable coefficients. Our object is to obtain an equivalent form with six squares, all with unit coefficients. Of course, if we admit imaginary transformations, all the coefficients can be made $+ 1$; but if we keep to the real domain, a certain number will equal $+ 1$ and a certain number $- 1$.

The fourth coefficient in (2) is a function of r alone; therefore we introduce a new variable R so that

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* Other fields in six dimensions exist and will be discussed elsewhere.

$$dR = \sqrt{\frac{2m}{r - 2m}}\,dr,$$

that is,

(3) $$R = \sqrt{8m(r - 2m)}.$$

Our form then becomes

(4) $$ds^2 = -\,dx^2 - dy^2 - dz^2 - dR^2 + \frac{R^2}{R^2 + 16m^2}\,dt^2.$$

The last term is of course not the square of an exact differential, and our general theorem on five-flats shows that we cannot obtain a form with five unit squares.

Taking the last two terms of (4), which involve merely the two variables R and t, we can, by the usual theory of surfaces, replace their combination

(5) $$dS^2 = -\,dR^2 + \frac{R^2}{R^2 + 16m^2}\,dt^2$$

by a sum of three unit squares. We readily find

(6) $$dS^2 = dX^2 + dY^2 - dZ^2,$$

where

(7) $$X = \frac{R \sin t}{\sqrt{R^2 + 16m^2}}, \qquad Y = \frac{R \cos t}{\sqrt{R^2 + 16m^2}},$$
$$Z = \int \sqrt{1 + \frac{256m^4}{(R^2 + 16m^2)^3}}\,dR.$$

The form (4) may now be written

(8) $$ds^2 = -\,dx^2 - dy^2 - dz^2 + dS^2;$$

but of course it must be remembered that while dx, dy, dz are exact differentials, dS is not. In fact dS is the distance element of the auxiliary surface defined by (7), where X, Y, Z (or rather X, Y, iZ) are cartesian coordinates in an auxiliary three-flat, and R and t are gaussian coordinates on this surface. This transcendental surface (involving a hyperelliptic integral), which apparently has escaped notice, we shall designate by (S). It is obviously a surface of rotation. To really obtain a form with all exact differentials we substitute (6) in (8). This gives our

THEOREM. *The exact solar field may be written with six squares of exact differentials*

(9) $$ds^2 = -\,dx^2 - dy^2 - dz^2 + dX^2 + dY^2 - dZ^2,$$

where X, Y, Z are defined by equations (7) *in conjunction with* (3) *and* (2′), *so that they are known functions of the world coordinates x, y, z, t.*

Thus if we consider a flat space with six rectangular coordinates x, y, z, X, Y, Z (or rather, on account of the minus signs in (9), with coordinates* ix, iy, iz, X, Y, iZ), *the finite representation of the solar field is a curved manifold* M_4 *of four dimensions* (*hypersurface*) *situated in a six flat and defined by* (7) *with* (3) *and* (2′).

If on this M_4 we take the parameter t equal to zero we obtain a submanifold M_3 which is peculiarly simple. From (7) we have $X = 0$, therefore the distance element of M_3 is, from (9),

$$-dx^2 - dy^2 - dz^2 + dY^2 - dZ^2, \tag{10}$$

where

$$X = \frac{R}{\sqrt{R^2 + 16m^2}}, \qquad Z = \int \sqrt{1 + \frac{256m^4}{(R^2 + 16m^2)^3}}\, dR. \tag{10′}$$

This shows that M_3 is in the five-flat (x, y, z, Y, Z). But going back to the form (4), and substituting $dt = 0$, we have as an equivalent element of M_3,

$$-dx^2 - dy^2 - dz^2 - dR^2; \tag{11}$$

or, changing all the signs, we use as the final form

$$d\sigma^2 = dx^2 + dy^2 + dz^2 + dR^2, \tag{12}$$

where R is defined by (3). This is obviously an M_3 in a four-flat. It is easily seen to be a rotation manifold. In particular if we put $z = 0$, we have

$$dx^2 + dy^2 + dR^2, \tag{13}$$

which defines a paraboloid obtained by rotating the parabola

$$R = \sqrt{2m(r - 2m)}, \tag{13′}$$

supposed drawn in the (r, R) plane, about its directrix (the axis of R). This simple result, defining the geometry in a plane through the sun, was first obtained by FLAMM.† It illustrates *the spatial measurements* for a given value of the time parameter t.

The surface (S) defined by (6) and (7) is obviously entirely different, and illustrates *the relation of the space and time measurements*. It is also a surface of rotation, but the generating curve is transcendental. From (7), we find

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* In Weyl's terminology this would be described by saying that the flat space here employed has four negative and two positive dimensions. The six space is not properly euclidean, but what Hilbert (*Göttinger Nachrichten*, 1915, 1917) would term pseudo-euclidean.

† L. Flamm, "Beiträge zur Einsteinschen Gravitationstheorie," *Physik. Zeitschr.*, Vol. 25 (1916), p. 163. A simple exposition is given in Weyl, "Raum, Zeit, Materie," 3d edition (1919), p. 223.

$$Z = \frac{1}{2}\int \sqrt{\frac{(1-P)^3 + 16m^2}{P(1-P)^3}}\, dP, \tag{14}$$

where

$$P = p^2 = \frac{R^2}{R^2 + 16m^2} = \frac{r - 2m}{r}. \tag{14'}$$

If we take p as abscissa and Z as ordinate we have the required curve. Rotation about the axis of ordinates (with Z replacement by iZ) produces the surface (S). The time parameter t is represented by the angle of rotation.

Flamm's paraboloid, defined by (13′), ***and the new surface, defined by*** (14), ***thus supplement each other, and together characterize the metric of the solar field.***

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